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Abstract

This article shows how Harmonic Analysis on matrix groups can solve specific problems of Multivariate Statistical Analysis. In particular are studied the properties of zonal polynomials, seen as spherical functions of a *classical* Gelfand pair. A duality formula allows to write the zonal polynomials as spherical functions of a *finite* Gelfand pair. Finally a combinatorial technique for the calculation of the zonal polynomials shows that these are the analogous, in the non-central Wishart distribution, of power in central Wishart distribution.

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GELFAND PAIRS AND ZONAL POLYNOMIALS IN MULTIVARIATE STATISTICAL ANALYSIS

Mauro Pagliacci

Introduction. — In some topics on non-central Multivariate Statistical Analysis appear mathematical objects that require specific tools to be suitably studied. The example which gave the inspiration for this work is the non-central Wishart distribution. By the works of A.T. James e A.G. Constantine ([J1,J2,Co]) is immediately evident the role of Harmonic Analysis space on symmetric matrices to compute of the density function of of non central Wishart distribution. In 1980 L.K. Kates, gave his Ph.D. thesis [Ka], on this subject. The approach of Kates makes use of Harmonic Analysis in a much more explicit way.

Kates works on *zonal polynomials*. From the point of view of the Harmonic Analysis, they are the spherical functions of a given Gelfand pair, i.e., in some sense, a generalization of the exponential function. From the point of view of the Multivariate Statistical Analysis the zonal polynomials appear in the density function of the non central Wishart distribution. Muirhead in [M], introduces the zonal polynomials by a recursion of symmetric functions. A different approach is given by A. Takemura in [T], using an original method, in the Muirhead spirit. All the above mentioned methods are rather laborious, in particular if the order of the polynomial is not very low. Making use of the Harmonic Analysis on symmetric matrices, the zonal polynomials appear in a very natural way.

In the present paper we give a short survey of the distributions of the multivariate statistical analysis and we introduce the topics to understand the Hua-Schmid-Takeuki theorem (Theorem 3.6). This theorem gives fundamental informations on the representations on the space of polynomials of a matrix argument random variable, allows to introduce the zonal polynomials and to study their main properties. Our approach is taken indirectly by the books of L.K. Hua [Hu] and R.H. Farrel [Fa].

The zonal polynomial are seen also as spherical functions of a classical infinite Gelfand pair. The duality formula of A.T. James, allows to reconstruct the spherical functions of the infinite Gelfand pair, given the spherical functions of a finite Gelfand pair, very easier to calculate. In this work is given a new proof for the duality formula.

We give also a combinatorial method to compute the coefficient of the zonal polynomials arising from the construction of a graph whose vertices are equivalence classes of the quotient group of the permutation group with respect to the *wreath product* with a subgroup of the permutation group. To calculate the zonal polynomials is also given the Kates integral formula.

For more details on the representations the symmetric groups probability and statistics, see the book of P. Diaconis [P]. Also the paper [FS] gives an interesting contribution to the combinatorics of zonal polynomials.

The present work is a survey of different point of views making origin by several seminars given at the Strasbourg University "Louis Pasteur" and the Rome University "La Sapienza".

1. The normal multivariate distribution. — Given a matrix A let us denote by A' his transpose. Let X be a *m-dimensional random variable*, i.e. a *random vector* $X = (X_1, X_2, \ldots, X_m)'$ whose components are the random variables $X_1, X_2, \ldots, X_m$. The *mean* (or *expected vaue*) of X is defined as

$$E(X) = (E(X_1), \ldots, E(X_m))',$$

where $E(X_i)$ is the mean of X_i, calculated with respect to the marginal distribution of X_i. In a more general setting, if $Z = (z_{ij})$ is a random $q \times q$ matrix, $E(Z)$ is the matrix whose element ij is $E(z_{ij})$ We remark that, if B, C, and D are matrices, respectively $m \times q$, $q \times n$ and $m \times n$, we have:

$$E(BZC + D) = BE(Z)C + D. \tag{1}$$

Indeed the element ij of $E(BZC + D)$ can be written as:

$$E(\sum_{g,h} b_{ih} z_{hg} c_{gj} + d_{ij}) = \sum_{g,h} b_{ih} E(z_{hg}) c_{gj} + d_{ij}.$$

Set $\mu = E(X)$. We define *covariance matrix* of x the $m \times m$ matrix $m \times m$

$$\Sigma = \mathrm{Cov}(X) = E[(X - \mu)(X - \mu)'].$$

The (i, j) element of Σ is

$$\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)'],$$

i.e. the covariance between X_i and X_j. The (i, i) element is

$$\sigma_{ii} = E[(X_i - \mu_i)^2],$$

i.e. the variance of X_i. Therefore the diagonal elements of Σ are non-negative and Σ is symmetric.

We can proof that the set of covariance matrices is equal to set of non-negative definite matrices.

A matrix $m \times m$ A is called *non-negative definite* if, for every α in $\mathbb{R}^m$, with $\alpha \neq 0$, it follows that $\alpha' A \alpha \geq 0$. A matrix is *positive definite* if, for every α in $\mathbb{R}^m$, with $\alpha \neq 0$, it follows that $\alpha' A \alpha > 0$.

The following lemma hols:

Lemma 1.1. — *The $m \times m$ matrix Σ is a covariance matrix if and only if is non-negative definite.*

Proof: Let Σ be the covariance matrix of a random vector X such that $E(X) = \mu$. Then, for each α in $\mathbb{R}^m$,

$$\mathrm{Var}(\alpha' X) = E[(\alpha' X - \alpha' \mu)] = E[(\alpha'(X - \mu))^2].$$

$\alpha'(X-\mu)$ is a scalar, then it coincides with his trasposted, so we can write:

$$\mathrm{Var}(\alpha' X) = E[\alpha'(X - \mu)(X - \mu)'\alpha] = \alpha' \Sigma \alpha \geq 0. \tag{2}$$

Therefore Σ is non-negative definite.

We suppose, conversely, that Σ is non-negative definite and of rank r with $r \leq m$. A matrix $m \times m$ non-negative definite and of rank r can be written as the product of a $m \times r$ of rank r and his transposed. In our case, the matrix $\Sigma = CC'$, with C matrix $m \times r$ of rank r. Let Y be a vector $r \times 1$ of independent random variables with mean 0 and covariance $\mathrm{Cov}(Y) = I$, where I is the identity matrix. Let $X = CY$. Then $E(X) = O$ and $\mathrm{Cov}(X) = E(XX') = E(CYY'C') = CE(YY')C' = CC' = \Sigma$. Therefore Σ is a covariance matrix. ∎

The inequality (2) implies that if the covariance matrix Σ of a random vector X is not-positive definite, then the components X_i of X are linearly dependent. Indeed, in this case there exists α in $\mathbb{R}^m$, with $\alpha \neq 0$, such that $\mathrm{Var}(\alpha' X) = \alpha' \Sigma \alpha = 0$. Therefore, with probability 1, $\alpha' X = k$, where $k = \alpha' E(X)$; i.e. X belongs to an hyperplane.

We consider now linear transformations and we study the behavior of covariance matrices.

Let X be a random vector $m \times 1$ of mean μ_X and covariance Σ_X. Sia $Y = BX + b$, where B is a matrix $k \times m$ and b is a vector $k \times 1$. By (1) we have:

$$\begin{aligned}
\mu_Y &= B\mu_X + b \\
\Sigma_Y &= [(Y - \mu_Y)(Y - \mu_Y)'] \\
&= E[(BX + b - (B\mu_X + b))(BX + b - (B\mu_X + b))'] \\
&= BE[(X - \mu_X)(X - \mu_X)']B' \\
&= B\Sigma_X B'.
\end{aligned}$$

To define the multivariate normal distribution need the following result:

Theorem 1.2. — *Let X be a $m \times 1$ random vector. Then the distribution of X is uniquely determined by the distributions of the linear functions $\alpha' X$, for each α in $\mathbb{R}^m$.*

Proof: The characteristic function of $\alpha' X$ is

$$\varphi(t, \alpha) = E(e^{it\alpha' X}).$$

In particular,

$$\varphi(1, \alpha) = E(e^{i\alpha' X}).$$

As a function of α, this is the characteristic function of X (i.e. the jointly characteristic function of the components of X). The result follows by the fact that a distribution in $\mathbb{R}^m$ is uniquely determined by his characteristic function. ∎

The $m \times 1$ random vector X has a *m-variate normal distribution* if, for each α in $\mathbb{R}^m$, the distribution of $\alpha' X$ is univariate normal.

The following properties hold. For a proof, see [M, MKB].

(a) If X has an m-variate normal distribution, then there exist $\mu = E(X)$ and $\Sigma = \text{Cov}(X)$ and the distribution of X is uniquely determined by μ and Σ.
Therefore if X has an m-variate normal distribution with mean μ and variance Σ, then X is $N_m(\mu, \Sigma)$.

(b) If X is $N_m(\mu, \Sigma)$, then the characteristic function of X is

$$\varphi_X(t) = \exp(i\mu' t - \frac{1}{2} t' \Sigma t).$$

(c) A linear transformation of a normal vector has normal distribution. In particular, if X is $N_m(\mu, \Sigma)$, B is $k \times m$ and b is $k \times 1$, then $Y = BX + b$ is $N_k(B\mu + b, B\Sigma B')$.

(d) if X is $N_m(\mu, \Sigma)$, then the marginal distribution of every subset of k components (with $k < m$) of X is k-variate normal.

(e) If X is $N_m(\mu, \Sigma)$ and X, μ and Σ are such that

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where X_1 and μ_1 are $k \times 1$ and Σ_{11} is $k \times k$. Then the subvectors X_1 e X_2 are independent if and only if $\Sigma_{12} = 0$.

Now we compute the density function of a random vector X in $N_m(\mu, \Sigma)$. We remark that if Σ is non-positive definite, and so singular, then X belongs to some hyperplane with probability 1. Therefore a density function for X, with respect to the Lebesgue measure on $\mathbb{R}^m$ may not exist. In this case we say that X has a singular normal distribution. If Σ is positive definite and so non singular, there exists the density function of X and it is easy to find, making use of the decomposition

$$X = CU + \mu,$$

where C is the matrix $m \times r$, of rank r, introduced in Lemma 1.1, $U = (U_1, U_2, \ldots, U_r)'$ is a vector of independent standard normal random variables and μ belongs to $\mathbb{R}^m$.

The following theorem holds (For a proof, see [M, MKB].

THEOREM 1.3 . — *If X is $N_m(\mu, \Sigma)$ and Σ is positive definite, then the density function of X is:*

$$f_X(x) = (2\pi)^{-m/2}(\det \Sigma)^{-1/2} \exp[-\frac{1}{2(x-\mu)'\Sigma^{-1}(x-\mu)}].$$

2. The Wishart distribution. — The di Wishart distribution is the multivariate generalization of the χ^2 distribution. Let $X_1, X_2, \ldots, X_n$ be independently distributed random vectors such that for every $i = 1, 2, \ldots, n$, X_i is $N_m(\mu_i, \Sigma)$. Then

$$W = \sum_{i=1}^{n} X_i X_i'$$

has a *Wishart distribution* with n degrees of freedom. W is said to be a Wishart matrix.

The Wishart distribution is *central* if $\mu_i = 0$, for every i. In this case we use the notation:

$$W \sim W_m(n, \Sigma).$$

Otherwise the distribution is *non central*, with notation:

$$W \sim W_m, (n, \Sigma, M)$$

where $M' = (\mu_1, \ldots, \mu_n)$.

The Wishart distribution can be obtained in a natural way taking n samples from a multivariate normal random vector X. We remark that, if $m = 1$, the distribution of $W_1(n, \sigma^2)$ is the same as $\sigma^2\chi^2$.

If A is $W_m(n, \Sigma)$, with $n \geq m$, then the *density function* of A is

$$\frac{1}{2^{\frac{nm}{2}}\Gamma_m(\frac{n}{2})(\det\Sigma)^{\frac{n}{2}}} \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}A\right)(\det A)^{\frac{n-m-1}{2}},$$

where A is a positive definite matrix and $\Gamma_m(\cdot)$ is the multivariate gamma function given by

$$\Gamma_m(a) = \int_{A>0} \text{etr(A)det} A^{a-\frac{m+1}{2}} dA,$$

with $\Re a > \frac{m-1}{2}$, where $A > 0$ denotes that A is a positive definite matrix and $\text{etr}(\cdot) = \exp \text{tr}(\cdot)$. We remark that, if $m = 1$, then $\Gamma_1(a) = \Gamma(a)$.

Some *non central* distributions in Multivariate Statistical Analysis can be obtained by integration on orthogonal groups or on *Stiefel manifolds*[1] with respect to an invariant measure which not can be computed in a closed form It is possible to see, for example, that if the $m \times m$ random matrix has $W_m(n, \Sigma)$ distribution, then the joint distribution of the eigenvalues $l_1, \ldots, l_m$ di A, for $n > m - 1$, is:

$$\frac{\pi^{\frac{m^2}{2}} 2^{-\frac{nm}{2}} (\det\Sigma)^{-\frac{n}{2}}}{\Gamma_m\left(\frac{m}{2}\right)\Gamma_m\left(\frac{n}{2}\right)} \prod_{i=1}^{m} l_i^{\frac{n-m-1}{2}} \prod_{i<j}^{m}(l_i - l_j) \int_{O(m)} \text{etr}\left(-\frac{1}{2}\Sigma^{-1}HLH'\right) dH,$$

with $l_1 > l_2 > \ldots > l_m > 0$, $L = \text{diag}(l_1, l_2, \ldots, l_m)$ and dH is the invariant measure on $O(m)$, normalized such that the volume of $O(m)$ is 1. We remark that in the above relation appear integrals over $O(m)$.

The above integral depends on Σ by its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. It is easy to see that this is a symmetric function of $l_1, l_2, \ldots, l_m$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$. The calculation of this integral can be given by a series expansion of the exponential and integrating term by term. In general this computation in very hard, unless you choose the " right " symmetric functions.

We can obtain some results comparing a univariate normal distribution with its corresponding multivariate distribution. Let $a = X'X$, where X is $N_m(\mu, I_n)$. Then the random variable a has a non central distribution $\chi_n^2(\delta)$, with $\delta = \mu'\mu$ and density function:

$$(3) \qquad \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \exp\left(-\frac{a}{2}\right) a^{\frac{n}{2}-1} \cdot \exp\left(-\frac{\delta}{2}\right) {}_0F_1\left(\frac{n}{2}, \frac{1}{4}\delta a\right),$$

where $a > 0$ and ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is the *generalized hypergeometric function*:

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot \ldots \cdot (a_p)_k}{(b_1)_k \cdot \ldots \cdot (b_q)_k} \frac{z^k}{k!},$$

[1] A *Stiefel manifold* is the set of $m \times m$ matrices H such that $H'H = I_m$, where I_m is the $m \times m$ identity matrix. If $n = m$ the Stiefel manifold is the orthogonal group $O(m)$; if $m = 1$, is the S_n, the unit sphere in $\mathbb{R}^n$.

where $(a)_k = a\,(a+1)\dots(a+k-1)$.

We remark that ${}_2F_1(a_1, a_2; b; z)$ is the classical hypergeometric function and that ${}_0F_1(b; z)$ is related with the Bessel function.

Let $A = Z'Z$ with Z in $N_m(M, I_n \otimes I_m)$, where $\otimes$ denote the *Kroneker product*, or the *direct product* of the matrices. We have that $M = E(Z)$ and the elements of the $n \times m$ matrix Z are independent and normally distributed with variance 1.

If $M = 0$, A has a distribution $W_m(n, I_n)$, with density function:

$$\frac{1}{2^{\frac{nm}{2}}\Gamma_m(\frac{n}{2})}\,\mathrm{etr}\left(-\frac{A}{2}\right)(\det A)^{\frac{n-m+1}{2}} \qquad (A > 0),$$

that can be reduced to the first part of (3), when $m = 1$.

If $M \neq 0$ then A is non central Wishart and, by invariance, A depends on M only by a matrix of "non centrality" $\Delta = M'M$. Moreover the Wishart non central density function can be bring back to (3), when $m = 1$. Therefore there is a natural generalization of the non central part

$$\exp(-\frac{\delta}{2})\,{}_0F_1\left(\frac{n}{2}; \frac{1}{4}\delta a\right)$$

changing δ with Δ ed a con A. The exponential $\exp(-\frac{\delta}{2})$ will be generalized to $\mathrm{etr}(-\frac{\Delta}{2})$ and the problem will be the generalization of the function ${}_0F_1$ with argument $\frac{1}{4}\delta a$ to a function with argument $\frac{1}{4}\Delta A$. For this purpose we will need of special function with matrix argument (see [H]).

We recall that:

$${}_0F_1(c; x) = \sum_{k=0}^{\infty} \frac{x^k}{(c)_k k!}.$$

The argument x must be replaced by a matrix and requires a generalization of x^k, in the matrix case. This is the role of the *zonal polynomials*, symmetrical polynomials with respect to the eigenvalues of X. The theory of zonal polynomials was developed by A.T. James and by A.G. Constantine in several work in the period 1960-1976 (see [J1,J2,Co]). The zonal polynomials can be introduced also in a direct way, as symmetric polynomials as in [M] e in [T], but this is very difficult. Another chance is to use the harmonic analysis methods on matrix spaces. In the following sections we survey this method, used also in [Fa, Ka].

3. Fisher product and Hua-Schmid-Takeuki Theorem. — Let $V = \mathbb{R}^m$ be a vector space with a scalar product (x, y). Let p be a polynomial on V. We associate to p a constant coefficients differential operator $p\left(\frac{\partial}{\partial x}\right)$ defined by:

$$p\left(\frac{\partial}{\partial x}\right)e^{(x,y)} = p(y)e^{(x,y)}.$$

The *Fisher product* is defined as:

$$< p,q >= p\left(\frac{\partial}{\partial x}\right)q(x)|_{x=0}.$$

We remark that, if we can write

$$p(x) = \sum_{|\alpha|\leq k} a_\alpha x^\alpha \qquad q(x) = \sum_{|\alpha|\leq k} b_\alpha x^\alpha,$$

in some orthonormal basis, then

$$< p,q >= \sum_\alpha \alpha!\, a_\alpha b_\alpha,$$

where:

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad x^\alpha = (x_1^{\alpha_1}, \ldots, x_m^{\alpha_m}), \quad |\alpha| = \alpha_1 + \ldots + \alpha_m.$$

Let $\mathcal{P}$ be the space of polynomials on V. $\mathcal{P}$ is a pre-Hilbert space of which the monomials are an orthogonal basis.

Let g be in $GL(V) = GL(m,\mathbb{R})$, the general linear group of $m \times m$ matrices with real coefficients and determinant different from zero. Let $\mathrm{End}(\mathcal{P})$ be the set od endomorphisms of $\mathcal{P}$. Consider the application

$$\pi\colon GL(V) \to \mathrm{End}(\mathcal{P}),$$

defined by

$$(\pi(g)p)(x) = p(g^{-1}x).$$

This is a representation of $GL(V)$ into the space $\mathcal{P}$.

Let $V = \mathrm{Sym}(m,\mathbb{R})$ be the space of $m \times m$ symmetric matices with real entries . We consider on V the inner product:

$$(x,y) = \mathrm{tr}\,(xy).$$

The group $G = GL(m,\mathbb{R})$ acts on V by $x \mapsto gxg'$. Let T be the group of lower triangular matrices with positive diagonal elements.

Proposition 3.1. — *Let t be an element of T. Then $x = tt'$ is positive definite symmetric matrix and, vice versa, every positive definite symmetric matrix x can be written in this way.*

Let $K = O(m)$ be the orthogonal group. We poof the following theorem.

THEOREM 3.2 (GAUSS DECOMPOSITION). —

$$G = TK,$$

in other words, every g in G can be written uniquely as $g = tk$, with t in T and k in K.

Proof: We set $x = gg'$. The matrix x is positive definite symmetric, therefore, by Proposition 3.1, there exists a unique t in T such that $x = tt'$. Denoting by $g_1 = t^{-1}g$, we have $g_1 g_1' = I$, and so g_1 is an orthogonal matrix, i.e. $g = tk$. ∎

We denote by $\Delta_k(x)$ the k-th *principal minor* of the matrix x (the determinant obtained choosing the first k rows end the first k columns of x.

PROPOSITION 3.3. — *Let X be a symmetric matrix and T be a lower triangular matrix. Then,*

$$\Delta_k(txt') = (t_{11} \cdots t_{kk})^2 \, \Delta_k(x).$$

Proof: We decompose the matrices x and t in the following way:

$$x = \begin{pmatrix} x_1 & x_2 \\ x_2' & x_3 \end{pmatrix}, \qquad t = \begin{pmatrix} t_1 & 0 \\ t_2 & t_3 \end{pmatrix},$$

where x_1 and t_1 are (k,k) matrices. We have:

$$txt' = \begin{pmatrix} t_1 x_1 t_1' & z_2' \\ z_2 & z_3 \end{pmatrix}.$$

Therefore

$$\Delta_k(txt') = \det(t_1 x_1 t_1') = \det(t_1)^2 \det x_1 = (t_{11} \cdots t_{kk})^2 \Delta_k(x)$$

∎

Let $\xi = (\xi_1, \ldots, \xi_m)$ be an element in $\mathbb{R}^m$. To ξ we associate a character of the triangular group T in the following way:

$$\chi_\xi(t) = t_{11}^{2\xi_1} t_{22}^{2\xi_2} \cdots t_{mm}^{2\xi_m}.$$

A polynomial P on Sym(m,$\mathbb{R}$) is *conical* if

$$P(txt') = \chi_\xi(t) P(x).$$

Let $\Delta_i(x)$ be the i-th principal minor of the matrix x. We suppose that $\xi_1, \ldots, \xi_m$ satisfy the relation $\xi_1 \geq \ldots \geq \xi_m$. Then the polynomial $P_\xi(x)$, defined by:

$$P_\xi(x) = \Delta_1^{\xi_1 - \xi_2}(x)\Delta_2^{\xi_2 - \xi_3}(x) \cdots \Delta_m^{\xi_m}(x),$$

by Theorem 3.2, is conical with respect to the character χ_ξ. Moreover P_ξ is of degree $|\xi| = \sum \xi_i$ and $P_\xi(I) = 1$.

The following theorems hold:

Proposition 3.4. — *Let P be a conical polynomial with respect to the character χ_ξ. Then $\xi_1, \ldots, \xi_m$ are integer numbers such that $\xi_1 \geq \xi_2 \geq \ldots \geq \xi_m \geq 0$ and*

$$P(x) = cP_\xi(x),$$

Where $c = P(I)$.

Proof: If x is a diagonal matrix with positive diagonal elements $a_1, a_2, \ldots, a_n$, then

$$P(x) = a_1^{\xi_1} \cdots a_n^{\xi_n}.$$

Because P is a polynomial, the numbers ξ_i are integers greater than or equal to zero. If $x = tt'$, with $t \in T$, we have $P(x) = c\chi_\xi(t)$, where $c = P(I)$ and, by Proposition 3.3, it follows

$$P(x) = c\Delta_1(x)^{\xi_1 - \xi_2}\Delta_2(x)^{\xi_2 - \xi_3} \cdots \Delta_\xi(x)^{\xi_n}.$$

We consider the following symmetric matrix:

$$x = \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & & & \cdots & & & 0 \\ 0 & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \alpha & \gamma & & & \vdots \\ \vdots & & & \gamma & \beta & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & 0 \\ 0 & & & \cdots & & & 0 & 1 \end{pmatrix}.$$

If $\alpha > 0$, $\beta > 0$ and $\gamma^2 < \alpha\beta$, then x is positive definite. Therefore,

$$P(x) = c\alpha^{\xi_i - \xi_{i+1}}(\alpha\beta - \gamma^2)\xi_{i+1}.$$

Since P is aa polynomial, this holds for all α β, γ. In particular, if $\beta = 0$,

$$P(x) = c\alpha^{\xi_i - \xi_{i+1}}(-\gamma^2)^{\xi_i+1}.$$

This implies $\xi_i - \xi_{i+1} > 0$. ∎

PROPOSITION 3.5. — *Let $\mathcal{P}$ a finite dimensional subspace of the space of the polynomials, invariant with respect to that action of G. If $\mathcal{P}$ is not zero, then there exists a conical polynomial on $\mathcal{P}$, different from zero.*

Proof: Let π be the representation of G on $\mathcal{P}$ defined by:

$$(\pi(g)p)(x) = p(g^{-1}xg^{-1\prime}).$$

By differentiation, we obtain a representation of the Lie algebra $\mathfrak{g} = M(m,\mathbb{R})$ of G. The operators $\pi(H)$, where H is a diagonal matrix, are self adjoint with respect to the Fisher product and commute. Therefore a simultaneous diagonalization is possible. Set

$$\mathcal{P}^{\lambda} = \{p \in \mathcal{P} \,:\, \pi(H)p = \lambda(H)p\},$$

where λ is a linear form:

$$\lambda(H) = \lambda_1 a_1 + \ldots + \lambda_m a_m,$$

$(a_1, \ldots a_m$ are the diagonal elements of $H)$. If $\mathcal{P}^{\lambda} = \{0\}$, λ is said to be a *weight* of the representation π and we have:

$$\mathcal{P} = \bigoplus_{\lambda} \mathcal{P}^{\lambda}.$$

Now we give an order to the space of linear form in the following way: $\lambda > \mu$ if $\lambda - \mu$ is positive on the set $\{H \,:\, a_1 < a_2 < \ldots < a_m\}$. Let $\lambda_{\max}$ be a dominant weight of the representation π, i.e. a maximal weight with respect to the order defined above and let p be polynomial in $\mathcal{P}^{\lambda_{\max}}$.

Let E_{ij} be the matrix with only one element different from zero in the i-th row and the j-th column equal to one. We have $[H, E_{ij}] = (a_i - a_j)E_{ij}$. Therefore,

$$\begin{aligned}\pi(H)\pi(E_{ij})p &= [\pi(H), \pi(E_{ij})]p + \pi(E_{ij})\pi(H)p \\ &= (a_i - a_j)\pi(E_{ij})p + \lambda_{\max}(H)\pi(E_{ij})p \\ &= [(a_i - a_j) + \lambda_{\max}(H)]\pi(E_{ij})p.\end{aligned}$$

If $i > j$, then $[\quad , \quad]$ is a linear form such that $\lambda > \lambda_{\max}$, therefore $\mathcal{P}^{\lambda} = \{0\}$ and $\pi(E_{ij})p = 0$. By linearity, we have $\pi(x)p = 0$ for each lower triangular matrix x with diagonal entries zero. The set of matrices is the Lie algebra $\mathfrak{n}$ of the subgroup N of the lower triangular matrices with diagonal entries equal 1.For matrix n of this type we have $\pi(n)p = p$. Let $\xi_1, \dots, \xi_m$ be the numbers defined by

$$\lambda_{\max}(H) = -2(\xi_1 a_1 + \dots, \xi_m a_m).$$

Then

$$\pi(\exp H) = \exp(-2(\xi_1 a_1 + \dots, \xi_m a_m)).$$

Every matrix t in T is the product of a matrix in N with a diagonal matrix, therefore:

$$\pi(t)p = t_{11}^{-2\xi_1} \cdots t_{mm}^{-2\xi_m} p,$$

So

$$p(txt') = \chi_m(t)p(x).$$

in other words, the polynomial p is conical. ∎

We give a characterization of some subspace of $\mathcal{P}$. Let $\xi = \xi_1, \dots, \xi_m$ be such that $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m \geq 0$. Denote by $\mathcal{P}_\xi$ the subspace of polynomials generated by $\pi(g)P_\xi$. We have $\mathcal{P}_\xi$ is a subspace of the space of homogeneous polynomials of degree $|\xi|$, therefore dim $\mathcal{P}_\xi$ is finite.

It is possible to proof that the only conical polynomials in $\mathcal{P}_\xi$ are multiples of P_ξ

The following very important theorem holds:

Theorem 3.6 (Hua - Schmid - Takeuki). —

(a) The subspaces $\mathcal{P}_\xi$ are irreducible.

(b) The subspaces $\mathcal{P}_\xi$ are pairwise orthogonal with respect to the Fisher product.

(c) If $\mathcal{P}_k$ is the subspace of homogeneous polynomials of degree k, then:

$$\mathcal{P}_k = \bigoplus_{|\xi|=k} \mathcal{P}_\xi.$$

Proof: (a) Let $\mathcal{Y}$ be an invariant subspace of $\mathcal{P}_\xi$ not reduced to $\{0\}$. By Proposition 3.5, $\mathcal{Y}$ contains a non zero conical polynomial proportional to P_ξ. Then $\mathcal{Y} = \mathcal{P}_\xi$.

(b) Let Q be the orthogonal projection on $\mathcal{P}_\xi$. We show that Q commute with the action of G. If f is a polynomial, Qf is the only element of $\mathcal{P}_\xi$ such that

$$f - Qf \perp \mathcal{P}_\xi.$$

Analogously, $Q\pi(g)f$ is the only element of $\mathcal{P}_\xi$ such that

$$\pi(g)f - Q\pi(g)f \perp \mathcal{P}_\xi.$$

Therefore,

$$\pi(g)f - \pi(g)Qf \perp \pi(g')\mathcal{P}_\xi = \mathcal{P}_\xi.$$

Then,

$$Q\pi(g)f = \pi(g)Qf.$$

We show that $\mathcal{P}_\xi$ and $\mathcal{P}_{\xi'}$ are orthogonal if $\xi \neq \xi'$. Indeed, $QP_{\xi'}$ is a conical polynomial with respect to the character $\chi_{\xi'}$. Since the only conical polynomials of $\mathcal{P}_\xi$ are multiples of P_ξ, it follows that $QP_{\xi'} = 0$. Moreover $Q\pi(g)P_{\xi'} = \pi(g)QP_{\xi'} = 0$, then $Q\mathcal{P}_{\xi'} = 0$

(c) The subspace

$$\mathcal{Y} = \mathcal{P}_k \ominus (\bigoplus_{|\xi|=k} \mathcal{P}_\xi)$$

is invariant. If it is not zero, contains a non zero conical polynomial (by proposition 3.4) and this is impossible. Then $\mathcal{Y} = \{0\}$. ■

4. The zonal polynomials. — For $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, with $\xi_1 \geq \xi_2 \geq \ldots \geq \xi_m \geq 0$, the *zonal polynomial (or sherical polynomial)* φ_ξ is defined by:

$$\varphi_\xi(x) = \int_K P_\xi(kxk')dk,$$

where dk is the normalized Haar measure of the orthogonal group K.

We remark that φ_ξ belongs to $\mathcal{P}_\xi$ and is K-invariant.

PROPOSITION 4.1. — *The K-invariant polynomials of the space $\mathcal{P}_\xi$ are proportional to φ_ξ.*

Proof: If f is a polynomial, we set

$$Q(f(x)) = \int_K f(kxk')dk.$$

Of course, Q is a projection of $\mathcal{P}_\xi$ on the subspace $\mathcal{P}_\xi^K$ of the K-invariant polynomials. Let g be an element on G. We have that $g = tk_1$, where $t \in T$ and $k_1 \in K$ (Theorem 3.2). Then,

$$[Q\pi(g^{-1})P_\xi](x) = \int_K P_\xi(tkxk't')dk = \chi_\xi(t)\varphi_\xi(x).$$

■

PROPOSITION 4.2. — *The zonal polynomials φ_ξ verify the following relation*

$$\int_K \varphi_\xi(gkxk'g')dk = \varphi_\xi(gg')\varphi_\xi(x).$$

Proof: In the proof of Proposition 4.1 we proved that:

$$\int_k P_\xi(gkxk'g')dk = P_\xi(gg')\varphi_\xi(x).$$

Replacing g with $k_1 g$ and integrating with respect to k_1, the result follows. ∎

Denote $\mathcal{P}^K$ the space of K-invariant polynomials. The following theorem holds:

Theorem 4.3. — *The spherical polynomials φ_ξ are an orthogonal basis for $\mathcal{P}^K$ and every polynomial p in this space can be written as*

$$p(x) = \sum_{\xi_1 \geq \cdots \geq \xi_m} a_\xi \varphi_\xi(x),$$

where $a_\xi =< p, \varphi_\xi > / < \varphi_\xi, \varphi_\xi >$.

Proof: The proof is an immediate consequence of the Theorem 3.6 and of the Proposition 4.1. ∎

The zonal polynomial $\varphi_\xi(x)$ depends only on the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of x, is homogeneous of degree $|\xi|$ and symmetric with respect to $\lambda_1, \lambda_2, \ldots, \lambda_m$. This allows to extend the definition of zonal polynomial to the case of square matrix. Indeed, if x is symmetric ed y positive definite, the eigenvalues of yx are the same of $y^{\frac{1}{2}}xy^{\frac{1}{2}}$. Therefore we can define $\varphi_\xi(yx) = \varphi_\xi(y^{\frac{1}{2}}xy^{\frac{1}{2}})$. Furthermore it is easy to see that φ_ξ is an eigenfunction of the Laplace-Beltrami operator of the space

$$\Omega \simeq GL(m, \mathrm{I\!R})/O(m).$$

A characterization of the zonal polynomials can be given by the spherical functions of the Gelfand pairs. The main definitions are given in a concise form; for more details, see [F]. Let G be a locally compact group and $C_c(G)$ the space of complex continuous functions with compact support on G. If K is a compact subgroup of G, we denote by $C_c^\natural(G)$ the space of functions f in $C_c(g)$ bi-K-invariant, i.e. such that, for every k and k' in K, it results $f(kxk') = f(x)$. $C_c(G)$ is a convolution algebra of which the space $C_c^\natural(G)$ is a sub algebra.

(G, K) is a *Gelfand pair* if the convolution algebra $C_c^\natural(G)$ is commutative.

We remark that, if G is a commutative group and K is the subgroup of the only neutral element, then (G, K) is a Gelfand pair. In particular, $(\mathrm{I\!R},\{0\})$ is a Gelfand pair.

A very useful criterion to identify a Gelfand pair is the following:

Proposition 4.4. — *(G,K) is a Gelfand pair if there exists an involutive automorphism ϑ of G such that, for every x in K, $x^{-1} \in K\vartheta(x)K$.*

For the proof, see [F, pag. 317].

If (G,K) is a Gelfand pair, a *spherical function* is a function φ continuous on G, bi-K-invariant and such that the application

$$f \mapsto \chi(f) = \int_G f(x)\varphi(x^{-1})dx$$

satisfies the relation

$$\chi(f * g) = \chi(f)\chi(g).$$

The spherical functions of the Gelfand pair ($\mathbb{R}$,$\{0\}$) are the exponential functions $\varphi(x) = \exp(\lambda x)$, where λ is a complex number. So the spherical function are the generalization of the exponential functions in the case of Gelfand pairs. An important characterization of the spherical function od a Gelfand pair is the following ([F, pag. 319]).

Proposition 4.5. — *A function φ, continuous on G, bi-K-invariant and non identically zero is spherical if and only if, for every $x, y \in G$ we have:*

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y).$$

In partucular, $\varphi(e) = 1$.

Another classical example of the Gelfand pair is $(GL(m,\mathbb{R}), O(m))$. The spherical polynomials defined above are the spherical functions of this Gelfand pair (corresponding to finite dimensional spherical representations).

5. The duality formula. — In this section we show that it is possible to find the zonal polynomials, i.e. the spherical functions of the Gelfand pair $(GL(m,\mathbb{R}), O(m))$, by knowing the spherical functions of a given *finite* Gelfand pair. We obtain the zonal polynomials by a development in terms of $\mathrm{tr}\,(x), \mathrm{tr}\,(x^2), \ldots$. The coefficients in this development are the spherical functions of a finite Gelfand pair. The formula was introduced by A.T. James in [J2] and reported by Kates in [Ka]. The proof provided here presents some innovations with respect to the original.

5.1. The finite Gelfand pair. — Set $|\xi| = k$. Let D be the set of pairs of $2k$ elements

$$\{(a_1a_2, a_3a_4, \ldots, a_{2k-1}a_{2k}) \,:\, a_i \in \{1, 2, \ldots, 2k\}\}.$$

We do not take into account neither the order of couples, neither the order in each pair. Let S_{2k} be the permutation group on $2k$ elements. S_{2k} is transitive on D. Denote by

$$o = (1\ 2, 3\ 4, \ldots, 2k\text{–}1\ 2k)$$

an element on D fixed as the origin and let H be the isotropy subgroup of o.

We have

$$H \simeq (S_2)^k \times S_k$$

and the cardinality of H is $2^k k!$.

H can be defined as the permutation group generated by S_k permuting the columns or changing the elements of a column in objects of type:

$$\begin{pmatrix} a_1, a_2, \ldots, a_k \\ a'_1, a'_2, \ldots, a'_k \end{pmatrix}.$$

The grup H, called the *wreath product* of S_2 andi S_k is denoted by $S_2 \sim S_k$.

We condire the quotient group:

$$H \backslash S_{2k} / H.$$

It is easy to see that there is a bijection between the partitions of k and the double cosets.[2] Indeed we consider a graph with $2k$ vertices and edges given by pair of subsequent vertices in the permutation. The number of vertices of the graph in every component (really a loop) is even. Therefore, dividing by 2 the number of vertices in every component, we obtain a partition of k.

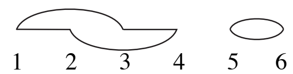


fig. 1

We explain the situation with an example. Take $k = 3$. The partition of 3 are: (3,0,0), (2,1,0) e (1,1,1). Consider S_6 and the element of S_6/H: $(1\ 3, 2\ 4, 5\ 6)$, as in fig.1. The graph is in bijection with the partition (2,1,0). Indeed, if we divide by 2 the length of every loop in the graph of fig. 1, we obtain the sequence (2,1,0).

[2] Recall that $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ is a *partition* of k if $\sum_{i=1}^{p} \xi_i = k$ with $\xi_1 \geq \xi_2 \geq \ldots \geq \xi_p \geq 0$.

In fig. 2 we have all the elements in every coset with the corresponding double cosets.

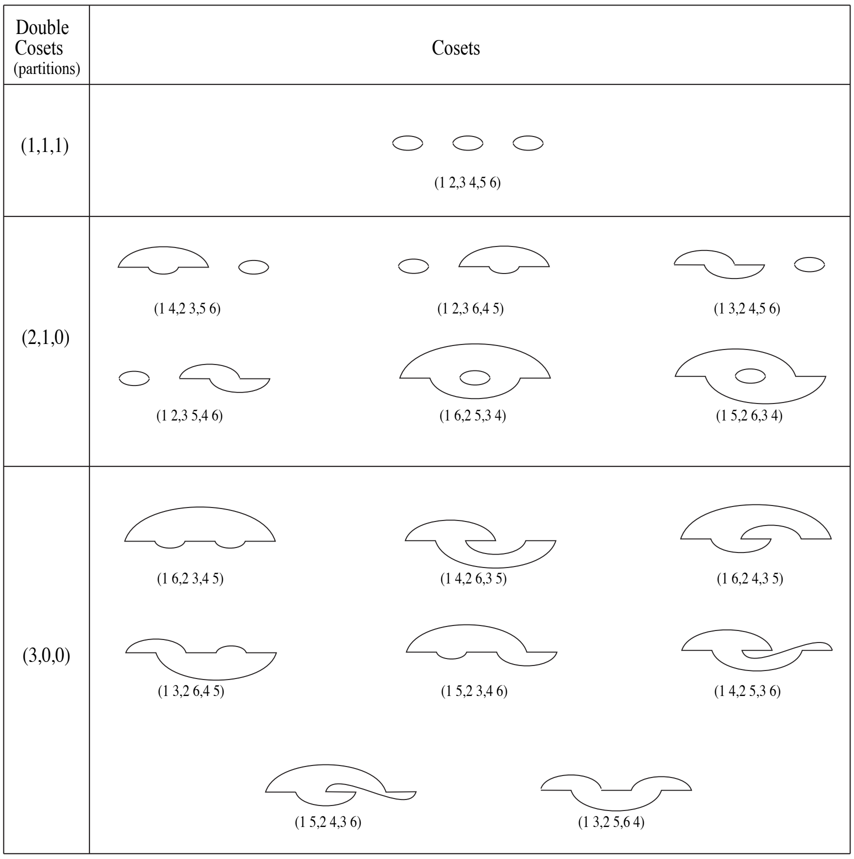


fig. 2

Proposition 5.1. — *(S_{2k}, H) is a Gelfand pair.*

Proof: Consider the element $\vartheta(\sigma) = \sigma$ in S_{2k}. Then $\sigma^{-1} \in H\sigma H$, therefore, by Proposition 4.4 the result follows ∎

5.2. The duality formula. — The duality formula allows to obtain the zonal polynomials, i.e. the spherical functions of the Gelfand pair $(GL(m, \mathbb{R}), O(m))$, by the spherical functions of the finite Gelfand pair (S_{2k}, H). The following Theorem holds.

Theorem 5.2 (Duality formula). —

$$\varphi_\xi(x) = \frac{1}{(2k)!} \sum_{\nu \in H \backslash S_{2k}/H} n_\nu \psi_\xi(\nu) \mathrm{tr}\,(x^{\nu_1}) \ldots \mathrm{tr}\,(x^{\nu_k}),$$

where n_ν is the number of cosets in every double coset ψ_ξ is the spherical *function of the Gelfand pair (S_{2k}, H) and $(\nu_1, \ldots, \nu_k)$ is the partition of k given by ν.*

The duality formula can be understood also by the spherical Fourier transform on D of a given function.

We introduce the following notation. Let U be the tensor product of $2k$ copies of $\mathbb{R}^m$, i.e. $U = \bigotimes_{2k} \mathbb{R}^m$,and consider the representation π of $GL(m, \mathbb{R})$ on U,

$$\pi \;:\; GL(m, \mathbb{R}) \to \mathcal{L}(U),$$

where $\mathcal{L}(U)$ is the set of endomorphisms of U. The representation is defined by:

$$\pi_g(v_1 \otimes \ldots \otimes v_{2k}) = gv_1 \otimes \ldots \otimes gv_{2k}, \qquad \text{with } g \in GL(m, \mathbb{R}).$$

We consider also the representation τ of S_{2k} on U,

$$\tau \;:\; S_{2k} \to \mathcal{L}(U),$$

defined by:

$$\tau_\sigma(v_1 \otimes \ldots \otimes v_{2k}) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(2k)}, \qquad \text{with } \sigma \in S_{2k}.$$

The representations π_g and τ_σ commutes, i.e. $\pi_g t_\sigma = \tau_\sigma \pi_g$ (see [M,W]). Set

$$a = (\operatorname{vec} I_m)^{\otimes k} = (\sum_{i=1}^{m} e_i \otimes e_i)^{\otimes k},$$

where (e_i) is the canonical base in $\mathbb{R}^m$ and $\operatorname{vec}(I_m)$ is the vector obtained by the sequence of columns of the identity $m \times m$ matrix I_m.

Consider the function:

$$f_g(\sigma) = (a, \pi_g \tau_\sigma a).$$

It is easy to see that $\pi_h a = a$, with $h \in O(m)$ and $\tau_\sigma(a) = a$, with $\sigma \in H$. Therefore f, as a function of g is $O(m)$-invariant and, as a function of σ is H-invariant.

The *spherical Fourier transform* of $f_g(\sigma)$ with respect to σ is given by:

$$\widehat{f}_g(\xi) = \text{const.} \sum_{\dot\sigma \in S_{2k}/H} \psi_\xi(\dot\sigma) f_g(\dot\sigma).$$

We remark that $f_g(\dot\sigma)$ is H-invariant.

We obtain the proof of the duality formula by the following formulas.

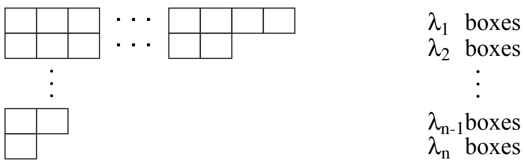
$$(A) \qquad f_g(\dot\sigma) = \operatorname{tr}(gg'^{\nu_1}) \ldots \operatorname{tr}(gg'^{\nu_k}),$$

$$(B) \qquad \widehat{f}_g(\xi) = \varphi_\xi(gg').$$

The proof of (A) can be obtained by direct calculation (see [Ka, lemma 5]). To proof (B) we need 3 lemmas.

LEMMA 5.3. — *The irreducibles components of π and τ are parameterized by partitions λ of $2k$.*

Proof: Let λ be a partition of $2k$. To λ corresponds an irreducible representation of S_{2k}. We associate to λ a *Young tableau* in the following way. If $\lambda = \lambda_1, \dots, \lambda_n$, with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, then the Young tableau is



The *Young diagram* is obtained associating at every box of a Young tableau an integer number between 1 and $2k$. To a Young diagram it is possible to associate the *Young idempotent* i_λ in the following way. Set $\mathcal{R}$ and $\mathcal{C}$ the subgroups of S_{2k} leaving globally unchanged, respectively, the rows and the columns of the Young diagram (the order is not considered). Then the Young idempotent is given by:

$$i_\lambda = \sum_{r \in \mathcal{R}, c \in \mathcal{C}} \varepsilon(c) \tau_r \tau_c,$$

where $\varepsilon(c)$ is the signum of the permutation c. Recall that, if G is a finite group with elements $g_1, \dots, g_m$, the group algebra of G is $A_G = \{a = \sum_{k=1}^m a(g_k) g_k\}$, where $a(g_k)$ are complex numbers. The lemma follows by the following result ([C,NS]). To any Young tableau identified by partition λ we associate a Young diagram giving the Young idempotent i_λ of the group algebra $A = A_{S_{2k}}$. Then $I^\lambda = Ai_\lambda$ are invariant subspace with respect to the left regular representation $\tilde{T}$ of S_{2k} and the restrictions $\tilde{T}^\lambda$ of the representation $\tilde{T}$ to I^λ are, of every λ a complete system of irreducible representations of S_{2k}.

In [W, Theorem 4.4.D] is proved that $i_\lambda(U)$ is an irreducible invariant subspace for π. Then the irreducible components of π are parameterized by the partitions of $2k$ and every invariant subspace is a direct sum of subspaces of this type. ∎

Write now the decomposition for π and τ in irreducible component,

$$\pi = \bigoplus_{|\lambda| = 2k} \pi^\lambda, \quad \tau = \bigoplus_{|\lambda| = 2k} \tau^\lambda,$$

and consider the characters

$$\chi_\lambda(g) = \mathrm{tr}\,\pi_g^\lambda \quad \text{e} \quad \chi_\lambda^*(\sigma) = \mathrm{tr}\,\tau_\sigma^\lambda.$$

Taking the group

$$S_{2k} \times GL(m, \mathbb{R}),$$

and the product of the representations $R = \pi\tau$, we obtain (see [M, pag. 107])

$$R = \bigoplus_{|\lambda|=2k} (\pi^\lambda \otimes \tau^\lambda).$$

Therefore U can be decomposed as

$$U = \bigoplus_{|\lambda|=2k} U_\lambda,$$

Where $U_\lambda = U_{\pi^\lambda} \otimes U_{\tau^\lambda}$.

Fix λ and consider the projector

$$T^\lambda = \frac{1}{(2k)!} \sum_\sigma \chi_\lambda^*(\sigma)\tau_\sigma.$$

It is easy to see that $(T^\lambda)^2 = T^\lambda$ and $T^\lambda(U) = U_\lambda$. Taking $\sigma = e$, R becomes representation of $GL(m, \mathbb{R})$ and we can write

$$U^\lambda = U_{\pi^\lambda} \oplus \ldots \oplus U_{\pi^\lambda} \quad (d_\lambda = \dim U_{\tau^\lambda} \text{ times}).$$

Lemma 5.4. — *If $\lambda = 2\xi$, with ξ partition of k, the representation $\pi^{2\lambda}$ admits a cyclic vector and we have:*

$$\varphi_\xi(gg') = \int_{O(m)} \chi_{2\xi}(gh)dh.$$

Proof: See [F]. ■

Set:

$$E = \int_{O(m)} \pi_h dh, \qquad E^* = \frac{1}{2^k k!} \sum_{\rho \in H} \tau_\rho.$$

Lemma 5.5. —

$$\mathrm{tr}\,(T^\lambda \pi_g E^*) = \chi_\lambda(g).$$

Proof: $\mathrm{tr}\,(T^\lambda \pi_g E^*) = \mathrm{tr}\,(\pi_{g|\mathcal{P}_\lambda \otimes \{\varphi_\lambda^*\}}) = \chi_\lambda(g)$. ■

The Lemma 5.3 e 5.4 imply that

$$\varphi_\lambda(gg') = \operatorname{tr}(T^\lambda \pi_g EE^*) = \operatorname{tr}(\pi_{g|\{a_\lambda\}}) = (\pi_g a_\lambda, a_\lambda),$$

where $\{a_\lambda\}$ is the subspace generated by a_λ.

Since $a_\lambda = \tau^\lambda a$, we have:

$$\varphi_\lambda(gg') = (\pi_g T^\lambda a, t^\lambda a) = (\pi_g (\tau^\lambda)^2 a, a) = (\pi_g T^\lambda a, a).$$

Replacing T^λ with its value, we can write:

$$\begin{aligned}
\varphi_\lambda(gg') &= \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} \chi_\lambda^*(\sigma)(\pi_g \tau_\sigma a, a) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} \chi_\lambda^*(\sigma) f_g(\sigma) \\
&= \frac{1}{(2k)! 2^k k!} \sum_{\dot\sigma \in S_{2k}/H} \sum_{\rho \in H} \chi_\lambda^*(\rho\sigma) f_g(\dot\sigma) \\
&= \frac{1}{(2k)! 2^k k!} \sum_{\dot\sigma \in S_{2k}/H} \psi_\lambda(\dot\sigma) f_g(\dot\sigma) \\
&= \frac{1}{(2k)!} \sum_{\nu \in H \backslash S_{2k}/H} n_\nu \psi_\lambda(\nu) f_g(\nu).
\end{aligned}$$

∎

Therefore the proof of the duality formula is complete.

The duality formula allows us to calculate the zonal polynomials, as an immediate application of the so called *character formula*. Ideed, if (G, K) is a Gelfand pair and Γ^i are irreducible subspaces of G with characters χ_i, then

$$\varphi_i(g) = \int_K \chi_i(gh) dh,$$

where φ_i is a spherical function in Γ^i. Taking the Gelfand pair (S_{2k}, H), the above formula give:

$$\psi_\lambda(\sigma) = \frac{1}{2^n n!} \sum_\mu \chi_\lambda(\sigma\mu).$$

But, the characters of the symmetric group are well known (see [L]), therefore this formula gives the spherical functions of the finite Gelfand pair.

We remark that the inversion formula of the spherical Fourier transform allows us to write:

$$f_g(\sigma) = \sum_\xi d_\xi \psi_\xi(\sigma) \varphi_\xi(gg'),$$

where $d_\xi = \dim U_{\tau^\xi}$.

6. Combinatorial computation of zonal polynomials. — Kates [Ka] make use of a normalization for zonal polynomials different from our. In our normalization we have $\varphi_\xi(I) = 1$, in that of Kates, really the most used in multivariate statistical analysis ([J1,J2,Mu]), the zonal polynomials are $z_\xi(x) = (\mathrm{tr})^m$ + other terms.

We show an algorithm that gives the coefficients $n_\nu \psi_\xi(\nu)$ (in the Kates normalization). To be more explicit, we consider a particular case, finding the coefficients of the third order zonal polynomial. Take $n = 3,\ k = 3,\ H = S_2 \sim S_3$ e $S_6/S_2 \sim S_3$.

We construct a graph whose vertices are the cosets $\{H\sigma\ :\ \sigma \in S_6\}$. In our example the vertice are the elements in fig.2. To plot the edges, it needs to fix a double coset, i.e. a partition of 3, for example (2,1,0). Then $[H\sigma_1, H\sigma_2]$ is an edge if $(H\sigma_1)(H\sigma_2)^{-1}$ is a permutation in the fixed double coset. We obtain in this way the following graph (fig.3)

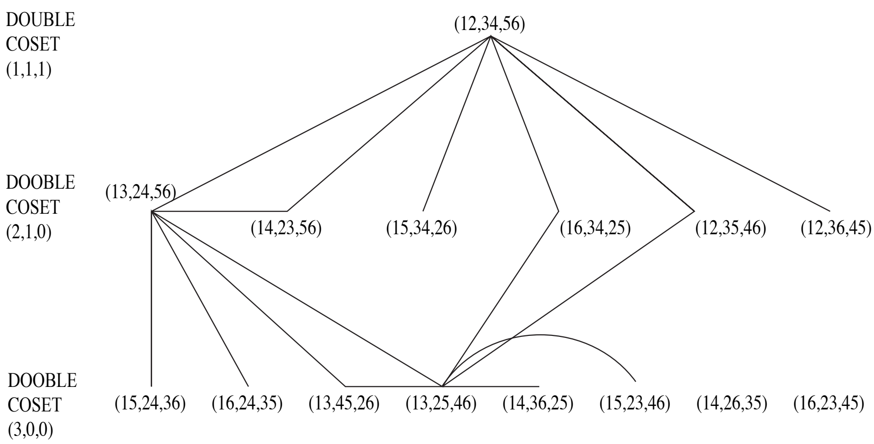


fig. 3

By the present graph, in which are plotted all the edges with a vertex on (12,34,56), (13,24,56) e (13,25,46), we can construct a matrix in the following way. Fix an element in each double coset, for example $(12, 34, 56) \in (1, 1, 1)$, $(13, 24, 56) \in (2, 1, 0)$ e $(13, 25, 46) \in (3, 0, 0)$. Let 1,2 and 3 be, respectively, the double cosets (1,1,1,), (2,1,0) e (3,0,0).The component $(C_{(2,1)})_{(\gamma,\beta)}$ in row γ and in column β is equal to the number of edges having a vertex on a fixed element of the double coset β and the other vertex on any elements of the double coset γ. Thereforene, in our example, we have:

$$C_{(2,1)} = \begin{pmatrix} 0 & 1 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 3 \end{pmatrix}.$$

The eigenvalues of this matrix are 6, 1, -3 and the eigenvectors (1,6,8), (1,1,-2) e (1,-3,2). In this case the eigenvectors are different, therfore we need not determine the matrices $C_{(1,1,1)}$ e $C_{(3,0,0)}$. Otherwise we must take the common eigenvectors, which gives the coefficient of the third order zonal polynomial in the basis $\mathrm{tr}\,(x)^3$, $\mathrm{tr}\,x\,\mathrm{tr}\,(x)^2$, $\mathrm{tr}\,(x^3)$. Ordering the basis (see [J2, Appendix]), with that above normalization:

$$\begin{aligned} z_{(1,1,1)} &= \mathrm{tr}\,(x)^3 + 6\mathrm{tr}\,(x)\,\mathrm{tr}\,(x^2) + 8\mathrm{tr}\,(x^3) \\ z_{(2,1,0)} &= \mathrm{tr}\,(x)^3 + \mathrm{tr}\,(x)\,\mathrm{tr}\,(x^2) - 2\mathrm{tr}\,(x^3) \\ z_{(3,0,0)} &= \mathrm{tr}\,(x)^3 - 3\mathrm{tr}\,(x)\,\mathrm{tr}\,(x^2) + 2\mathrm{tr}\,(x^3). \end{aligned}$$

Actually the above matrix is the matrix of a convolution operator and the common bi-K-invariant eigenfunctions are spherical functions of the Gelfand pair.

If (G,H) is a Gelfand pair, the eigenfunctions of the convolution operator by a continuous function on G and bi-H-invariant are the spherical zonal functions

Let $g_1,\dots,g_m$ be a set of representative elements in $H\backslash G/H$. Define

$$\delta_{Hg_iH}(g) = \begin{cases} 1, & \text{if } g \in Hg_iH; \\ 0, & \text{if } g \notin Hg_iH. \end{cases}$$

As a convolution operator, this gives a basis for $C(H\backslash G/H)$, the space of complex functions on $H\backslash G/H$. The problem is to find the simultaneous eigenfunctions of the convolution operator with δ_{Hg_iH}.

Let α, β, $\gamma,\dots$ be double cosets and g_α, g_β, $g_\gamma,\dots$ representatives of double cosets. The set of linear combinations of functions $\delta_{Hg_\alpha H}$ is a convolution algebra. The duality formula says that the coefficients of $\mathrm{tr}\,(x)^{\nu_1}\,\mathrm{tr}\,(x)^{\nu_2}\dots$ in the development to obtain the zonal polynomials are the spherical functions of $(S_{2k}, S_2 \sim S_k)$ multiplied by n_ν, the number of cosets in every double coset. We have $n_\nu = |Hg_\nu H|/|H|$. Then we use the base $|Hg_\alpha H|^{-1}\delta_{Hg_\alpha H}$ instead of $\delta_{Hg_\alpha H}$ to avoid the multiplication by n_ν and we compute the matrix of the convolution operator by $|H|^{-1}\delta_{Hg_\alpha H}$ in the basis $\{|Hg_\alpha H|^{-1}\delta_{Hg_\alpha H}\}_\alpha$. This is the matrix C_α. The simultaneous eigenfunctions of this matrix are the coefficients in the development of zonal polynomials.

The same argument can be done also using the Laplace Beltrami operator of the graph given in fig. 3. If Δ is the Laplace Beltrami operator on the graph, we can write $\Delta = A - \Lambda$, where A is the adjacency matrix of the graph (i.e. $A_{xy} = 1$ if x and y belong to the same edge and $A_{xy} = 0$, otherwise) and Λ is the diagonal matrix whose element xx is $|E_x|$, E_x being the set of the vertices of the graph neighbors to x. It is also possible to write $\Delta = A - n_\alpha I$, where n_α is the number of cosets in the double coset α. Take $Hg_\beta H$. Then:

$$A\delta_{Hg_\beta H}(x) = \sum_{y\in E_x} \delta_{Hg_\beta H}(y) = \sum_{y\in H\backslash G} \delta_{Hg_\alpha H}(xy^{-1})\delta_{Hg_\beta H}(y)$$
$$= |H|\delta_{Hg_\alpha H} * \delta_{Hg_\beta H}(x).$$

Indeed $y \in E_x$ if and only if $xy^{-1} \in \alpha$ if and only $\delta_{Hg_\alpha H}(xy^{-1}) = 1$. We remark that the functions δ are defined on G instead of on $H\backslash G$. Then A is the convolution operator by $|H|^{-1}\delta_{Hg_\alpha H}$. Therefore we have characterized the basis for the double coset in terms of Laplace-Beltrami operator.

7. The integral formula. — The following result is an integral formula ofr the evaluation of zonal polynomials

Theorem 7.1 (integral formula). — *Let $E = M(m, \mathbb{R}) \simeq \mathbb{R}^{m^2}$ be the set of $m \times m$ matrices with real coefficients. Then:*

$$\varphi_\xi(x) = \text{cost}\int_E \exp(- \parallel \gamma \parallel^2)P_\xi(\gamma' x\gamma)d\gamma,$$

where $\gamma \in E$ and P_ξ is the conical polynomial defined by

$$P_\xi(x) = \Delta_1^{\xi_1-\xi_2}(x)\Delta_2^{\xi_2-\xi_3}(x)\cdots\Delta_m^{\xi_m}(x).$$

To prove the Theorem, we use *the polar decomposition* of E. Let Ω be the cone of positive definite symmetric matrices and E' the set:

$$E' = \{\gamma \in E \,:\, \det\gamma'\gamma \neq 0\}.$$

Polar Decomposition. — *For every γ in E', there exist a matrix h in $O(m)$ and a matrix r in Ω such that $\gamma = hr^{\frac{1}{2}}$*

Let $\gamma \in E$. It is possible to write:

$$\gamma = (\gamma\gamma')^{\frac{1}{2}}(\gamma\gamma')-\frac{1}{2}\gamma.$$

Set $r = \gamma\gamma'$ e $h = (\gamma\gamma')-\frac{1}{2}\gamma$. We have that $r \in \Omega,\ h \in O(m)$, then the polar decomposition is:

$$\gamma = r^{\frac{1}{2}}h.$$

Similarly it is possible to show that $\gamma = hr^{\frac{1}{2}}$. The Jacobian of the transformation is:

$$d\gamma = \frac{c_0}{2^m}(\det r)^{-\frac{1}{2}}drdh,$$

where $dr = \prod_{i\leq j} r_{ij}$ and dh is the normalized Haar measure on $O(m)$. It is possible to compute that

$$c_0 = \frac{\pi^{\frac{m^2}{2}}}{\Gamma_\Omega(\frac{m}{2})},$$

where Γ_Ω is the Gamma function on the cone Ω, given by the following Siegel integral:

$$\Gamma_\Omega(\xi) = \int_\Omega \exp(-\mathrm{tr}(r))P_\xi(r)(\det r)^{-\frac{m+1}{2}}dr.$$

Proof of Theorem 7.1. If $x \in \mathrm{Sym}(m, \mathbb{R})$, we need to calculate the integral

$$I_\xi(x) = \int_E \exp(- \parallel \gamma \parallel^2)P_\xi(\gamma' x\gamma)d\gamma.$$

Changing the variables $\gamma = \eta x^{-\frac{1}{2}}$ and taking, at the moment, $x \in \Omega$, we have:

$$\begin{aligned}
&\parallel \gamma \parallel^2 = \mathrm{tr}\,(x^{-1}\eta'\eta) = (x^{-1}, \eta\eta') \\
&P_\xi(\gamma x\gamma') = \pi_\xi(\eta'\eta) \\
&d\xi = (\det x)^{-\frac{m}{2}}d\eta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_\xi(x) &= (\det x)^{-\frac{m}{2}} \int_E \exp(-(x^{-1}, \eta\eta'))P_\xi(\eta\eta')d\eta \\
&= c_0 \int_\Omega\int_{O(m)} \exp(-(x^{-1}, h'xh))P_\xi(r)(\det r)^{-\frac{1}{2}}dhdr.
\end{aligned}$$

Now ve compute the integral on Ω. Since $(x^{-1}, hrh') = \mathrm{tr}\,(x^{-1}hrh') = \mathrm{tr}\,(h'x^{-1}hr) = (h'x^{-1}h, r)$, we can write:

$$\begin{aligned}
&\int_\Omega \exp(-(x^{-1}, h'xh))P_\xi(r)(\det r)^{-\frac{1}{2}}dr \\
&\qquad = \int_\Omega \exp(-(h'x^{-1}h, r))P_\xi(r)(\det r)^{-\frac{1}{2}}dr \\
&\qquad = \int_\Omega \exp(-(h'x^{-1}h, r))P_\xi(r)(\det r)^{\frac{m+1}{2}-\frac{1}{2}}(\det r)^{-\frac{m+1}{2}}dr \\
&\qquad = \int_\Omega \exp(-(h'x^{-1}h, r))P_{\xi+\frac{m}{2}}(r)(\det r)^{-\frac{m+1}{2}}dr \\
&\qquad = \Gamma_\Omega\left(\xi + \frac{m}{2}\right)P_{\xi+\frac{m}{2}}(h'xh).
\end{aligned}$$

Therefore,

$$I_\xi(x) = \frac{(\det x)^{-\frac{m}{2}} c_0}{2^m} \Gamma_\Omega\left(\xi + \frac{m}{2}\right) \int_{O(m)} P_{\xi+\frac{m}{2}}(h'xh)dh,$$

where

$$\int_{O(m)} P_{\xi+\frac{m}{2}}(h'xh)dh = \varphi_{\xi+\frac{m}{2}} = \varphi_\xi(x)(\det(x))^{\frac{m}{2}}.$$

Because

$$I_\xi(x) = c\Gamma_\Omega\left(\xi + \frac{m}{2}\right)\varphi_\xi(x),$$

the theorem is proved. ∎

The integral formula can be used to obtain the zonal polynomials of each order by a computer program.

Now we are able to understand the non central Wishart distribution. We define the following function on the space $\mathrm{Sym}(m, \mathbb{R})$:

$${}_0F_1(\beta, x) = \sum_\xi \frac{1}{< \varphi_\xi, \varphi_\xi >} \frac{\Gamma_\Omega(\beta)}{\Gamma_\Omega(\beta+\xi)} \varphi_\xi(x).$$

By the Laplace transform of ${}_0F_1(\beta, x)$, we can find that the non central Wishart distribution $W_m(n, \Sigma, \delta)$ has density function:

$$\exp\left(-\frac{1}{2}\mathrm{tr}\,\delta\right) {}_0F_1\left(\frac{n}{2}; \frac{1}{4}\delta^{\frac{1}{2}} x \delta^{\frac{1}{2}}\right) \frac{1}{2^{\frac{nm}{2}}\Gamma_\Omega(\frac{n}{2})} \exp\left(-\frac{1}{2}\mathrm{tr}\,x\right) (\det x)^{\frac{n}{2}-\frac{m+1}{2}}.$$

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